On the Contribution of Black, Scholes and Merton

A History of Option Pricing and Mathematical Finance

Frederik Amris
Department of Economics,
University of Copenhagen

August 31st, 2016

Abstract

In 1973, Fisher Black and Myrion Scholes published their famous paper "*The Pricing of Options and Corporate Liabilities*" in which they developed the Black-Scholes formula, used world-wide for option pricing. Later the same year, Robert Merton published his paper "*Theory of rational option pricing*", which generalized and refined Black-Scholes arguments. In this paper, we discuss the contribution of Black, Scholes and Merton in relation to the preceding history of option pricing and examine the core of their contribution. Also, we discuss the importance of their work with respect to the subsequent developments in mathematical finance.

Contents

1 Introduction

The *Sveriges Riskbank Price in Economic Sciences in Memory of Alfred Nobel* was in 1997 awarded jointly to Robert C. Merton and Myron S. Scholes for "*a new method to determine the value of derivatives*", [1]. The theory of option pricing are in general attributed to them for their papers, [2] and [3] respectively, both published in 1973. In the former, BS derived the famous Black-Scholes equation and in the latter, Merton established a rigorous framework, or in some sense a set of axoims, to be satisfied for any rational option pricing theory. Also, he gave Black and Scholes' solution to the option pricing problem a careful treatment with Occam's Razor.

This paper is on the joint contribution of Black, Scholes and Merton (BSM henceforth[1]). In what follows, we'll examine the history of asset pricing before 1973 and how subsequent work was influenced by their research. Also, we will argue that the main contribution is *not* the option pricing formula itself, but rather how it was derived, paving the way for a correct separation between mathematical finance and mean-variance optimization in financial economics established in the fifties. The formula, as we will see, had basically been derived prior to 1973, but these derivations had either unobservables, such as risk-aversion parameters, or lacked a satisfactory economic foundation. The purpose of this paper is therefore threefold. Firstly, to investigate whether people were "helpless" in pricing options (somewhat) correctly pre-1973. Secondly, to investigate how BSM's thinking was original, and lastly how, and if, their contribution was central to the subsequent developments in mathematical finance.

The paper is organized as follows. In section 2, the papers of Black-Scholes and Merton are discussed. Section 3 gives an account of the history of asset pricing. Section 4 concludes with a discussion.

2 The Black-Scholes-Merton Model of 1973

A derivative is a contract that derives it's value from an underlying asset. In this paper, it's implicitly understood that the underlying is a stock, but this need not be the case. Prior to 1973, the literature was primarily concentrated on stock options, which typically comes in two variants: European and American. A European call (put) option on a stock gives the option-owner the right to buy (sell) a stock at a certain price (called the strike)

[1]Often, the model is just referred to as the Black-Scholes model.

at a pre-specified future date, called the maturity date. The American counterpart gives, in addition, the right to exercise the option at any date prior to and including the maturity date.

2.1 Black and Scholes' Derivation of the Formula

In this paper, we discuss the European call option, since this was the focus of Black and Scholes. We focus mainly on the call option and not the put option, since their respective prices are closely related through what is known as the "put-call parity". Mathematically, the payoff of a call option can be represented as

$$\max(S_T - K, 0), \tag{1}$$

where S_t is the underlying price process, K the strike and T time to maturity of the option. The famous Black Scholes formula of [2] states, that the price of a call-option at time t with maturity at time T on a stock, S_t, is given by:

$$\begin{aligned} C(S_t, t) &= N(d_1)S_t - N(d_2)Ke^{-r(T-t)}, \\ d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right], \\ d_2 &= d_1 - \sigma\sqrt{T-t}, \end{aligned} \tag{2}$$

where $N(\cdot)$ is the cumulative distribution function of the standard normal distribution, r is a risk free interest rate, often considered to be the yield on a government bond, and σ is the volatility of the underlying stock process. A way to think about the volatility is as the annualized standard deviation of returns over an appropriate time-horizon. The stock price process is assumed to follow a geometric Brownian motion, such that the dynamics of the process is given by:

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{3}$$

Here, S_t is the price process of the underlying, μ is the drift of the underlying. W_t is known as a Wiener process, or Brownian motion[2]. The Brownian motion has the property that the increments, $W_t - W_{t-1}$, are normally distributed. Note, how the drift-paramter, μ, does not enter the BSM-formula. μ is closely related to the expected value of the underlying at a specific future point in time, and since it does not enter the formula,

[2]The uninitiated reader may think of this as the continuous time equivalent of a random walk.

investors assessment on the future does not matter for the option price according to the model. Black and Scholes derived the formula from the following parabolic differential equation, known as the Black Scholes equation[3]:

$$\frac{\partial V_t}{\partial t} + \frac{1}{2}\,\sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} + r S_t \frac{\partial V_t}{\partial S_t} - r V_t = 0 \tag{4}$$

$$V_T = \max(S_T - K, 0). \tag{5}$$

Here, V_t is a so-called self-financing trading strategy (or portfolio), i.e. a specification of how to divide funds between the underlying stock and a bank account, earning the interest rate, r, at a specific point in time, t. Equation (5) is a terminal condition that must be satisfied, and this corresponds to the payoff of the option. This equation lies at the heart of their main contribution and is essentially derived from the following argument, taken from BS, [2]:

> "If options are correctly priced in the market, it should not be possible to make sure profits by creating portfolios of long and short positions in options and their underlying stocks. Using this principle, a theoretical valuation formula for options is derived."

What they are saying here is the following. If one can form a (self-financing) portfolio, V_t, that under any circumstance gives the same payoff as the option at time T, then buying the option and selling this portfolio should not yield a profit. In other words, if it's possible to create a dynamic trading strategy that replicates the option, then the value of the option *must* equal the value of portfolio[4]. This an economic argument; If one is more expensive than the other, there is an arbitrage opportunity that people will utilize (the law of one price). Note, that the theory generalizes to other derivatives by replacing the terminal condition in (5).

To summarize, the price of the portfolio, V_t, that solves the partial differential equation in (4) gives the formula in (2) and hence the option price. In their paper, they derive this equation in two ways. First, they solve (4) by a clever change-of-variable method and secondly they derive it with an application of the CAPM model, to be discussed below. In fact, the latter derivation was effectively based on applying CAPM at every instant in infinitesimally small time-periods, see BS 1973 themselves, or for modern treatment, [8] by Daniel Duffie.

---

[3]Note, that we've adopted a modern notation, as in for instance Bjork, [24].

[4]The replication-argument requires rather strict assumptions, such as no transaction costs and that trading can be done continuously.

2.2 Merton's Extensions and Refinement

Mertons paper of 1973, entitled "Theory of rational option pricing", expands on the ideas of Black and Scholes. In fact, Black and Scholes attribute the idea that the return on a hedged position (i.e. a portfolio of the option and the replicating portfolio discussed above) becomes certain (and is therefore risk-free). The idea here is, that something that is risk-free must earn the risk-free return in the model (i.e. r), but this is just another way of saying that there can be no arbitrage opportunities. Merton's paper is divided in different sections in which he accomplishes different ends. For instance, he derives a set of theorems that he claims must be hold for any option pricing theory to be rational, but for our purpose, and perhaps most importantly, he generalizes the Black Scholes model to allow for stochastic interest rates, dividends and a more general functional form of the volatility, σ[5]. Merton also examines meticulously the assumptions made in [2]. He notes, that while Black and Scholes derives the option pricing formula as a special case of CAPM, their solution does not depend on any of it's endogenous parameters. Only the interest rate is needed, which is assumed exogenous. Therefore, he concludes that CAPM's assumptions on investors preference and general equilibrium is not needed for the formula to be true. In his 1973 paper, he writes (comments in brackets):

> "More to the point, it is only necessary to consider three assets to derive the BS formula [*i.e. the option, the stock and the interest rate*]. In connection with this point, although BS claim that their central assumption is the capital asset pricing model (emphasizing this over their hedging argument), their final formula depends only on the interest rate [...] and not on the total variance of the return on the common stock. It does not depend on the betas [...] or other asset's characteristics. Hence this assumption may be a 'red herring'."

In Merton's derivation, he goes on to show that the only necessary assumption needed, is that investors prefer more to less, which is summarized in his assumption 1 of [3]. In other words, the assumptions of CAPM are sufficient to derive the BSM formula, but not *necessary.*

This point is important, since it is the foundation of modern mathematical finance. It is effectively what separates financial economics from mathematical finance. The consequence is, as we shall discuss later, that it does not matter what we assume about the investors of the economy, and for this reason we may as well assume, that they are risk-neutral.

[5]In Merton's derivation, the volatility is at most allowed to be time-dependent.

3 A Brief History of Asset Pricing

This section is on the history of asset pricing with a focus on options, but such a treatment would not be complete without mentioning the major developments in financial economics, and how modern mathematical finance evolved to be a separate research area. Here, we focus only on the essentials, but broader treatments can be found in for instance [25], [26], [27] or [28].

3.1 Louis Bachelier and His "Théorie de la spéculation"

One of the earliest records of option pricing dates back the Louis Bachelier's PhD Thesis *"Theorie de la speculation"* from 1900. Bachelier's work was very influential in certain branches of mathematics and science, but was largely ignored by the economics establishment for 50 years after it's publication, until Paul Samuelson, among others, received a letter from a Jimmie Savage, an American mathematician and statistician, who had stumbled upon it and wanted to know if it was of any use the his economist acquaintances[6]. Bachelier's style was messy, with several errors, but nonetheless, he succeeded in deriving important mathematical results, of which he was only partially credited in the formal literature. From a mathematical perspective, he succeeded in describing other important relations, such as the famous "Fokker-Planck" and "Chapman-Kolmogorov" equations as well as the Brownian motion, encountered in section 2, used in finance and numerous other applications[7].

Bachelier effectively modeled the dynamics of the underlying stock as a Brownian motion. This implies, that

$$dS_t = \sigma dW_t \tag{6}$$

in modern notation. One dissatisfactory consequence of this is, that stock prices can attain negative values. He also managed to derive a closed form solution for the price of an option under this assumption, known today as the normal-model. The truly remarkable thing is, that Bachelier was able to derive both a closed-form formula for the option price and the (at the time) unknown mathematics necessary to derive it.

[6] This is discussed in [28], in which Paul Samuelson has written a very interesting foreword.
[7] In fact, Albert Einstein is often credited for formalizing the idea of a Brownian motion in 1905.

3.2 Financial Economics and the Capital Asset Pricing Model

Before discussing other research on option pricing, we will discuss some important developments in financial economics in the mid-20th century. In 1954, Kenneth Arrow and Gerard Debreu developed what was later to be known as Arrow-Debreu prices (or state-prices), in their paper, [11], of 1954. That is, the price of receiving 1 unit of numeraire currency in one "state of the world" and 0 in all others. This idea is closely related to the theory of risk-neutrality and martingale measures (to be discussed later). It can be shown, that a 1:1 correspondence exists between state-prices and the risk-neutral probabilities in discrete time models, such as the famous Binomial model of Cox, Ross and Rubinstein in [6] (1979).

Other important research comes from Modigliani and Miller, known for the famous "Modigliani Miller theorem"[8], that Merton [3] claims is used in Black and Scholes' argument in an "inter-temporal version". Perhaps more importantly, for the purpose of our investigation, they were also the first to formalize the notion of arbitrage-free pricing, which is the foundation of modern mathematical finance, presented in their paper of 1958, [12]. While these development has without a doubt had a profound impact of the thinking at the time, our main focus of this section is on the capital asset pricing model (CAPM), developed by Jack Treynor, John Litner and most notably William F. Sharpe in his 1964 paper, [10]. As mentioned, Black and Scholes derived the Black-Scholes equation in two ways, one of which was based on CAPM. The work on CAPM was effectively an extension of Markowitz' portfolio selection theory of 1952, [9], in which he realized that investors should not only care about the maximal expected return, but also on the variance of returns.

The CAPM model is an attempt to apply the ideas of general equilibrium theory to asset pricing. One major conclusion of the model is that assets should only yield an expected return in excess of some risk free rate, if the asset has systematic risk that cannot be eliminated by diversification. The CAPM equation is:

$$\mathbb{E}(r_i) = \alpha_i + r_f + \frac{\text{COV}(r_i, r_m)}{\text{VAR}(r_m)}\Big(\mathbb{E}(r_m) - r_f\Big) = \alpha_i + r_f + \beta_i\Big(\mathbb{E}(r_m) - r_f\Big) \tag{7}$$

where r_i is return on the asset, α_i is the excess return on the asset compared to the prediction of CAPM (effectively 0 according to the theory), r_f is the risk free rate (i.e.

[8]The theorem says, that under perfect market conditions, it does not matter how a firm chooses to finance it's activities through debt or equity.

government bonds) and β_i is the assets correlation with *market portfolio* (also called systematic risk) with return r_m. Hence, if an asset has a high beta, market-moves of a certain size is amplified for that particular asset. William Sharpe notes in [10], that the model has "highly restrictive and undoubtedly unrealistic assumptions", but goes on to justify it's relevance by stating that

> "The proper test of a theory is not the realism of it's assumptions but the acceptability of it's implications"

in the spirit of Milton Friedman's essay, [13], of 1953.

The important point from our perspective is, that the equation is the result of a partial equilibrium, in which a representative agent solves a mean-variance optimization. The agents preferences does not enter the CAPM equation explicitly, but the equation is a result of strict assumptions on these. Note also, that CAPM assumes that asset returns are normally distributed. This can never be the case for a call option, since the payoff is truncated, with unlimited upside and limited downside, and for this reason alone CAPM is not suitable for pricing options.

3.3 Option Pricing Theory before Black, Scholes and Merton

In this section, we'll examine some notable research in option pricing post Bachelier and prior to Black, Scholes and Merton[9].

In [2], Black and Scholes mentions several attempts, prior to their own publication, of deriving option pricing formulas. The earliest was [17], of Sprenkle in 1961. His formula, that BS included in their paper, resembles the BSM-formula a lot, but with the important difference, that Sprenkle's formula included two unobservable parameter related to the expected value of future asset prices. Sprenkle was also the first to model stock prices using the geometric Brownian motion, also used by BSM in 1973. James Boness derived a similar formula in 1964 in his paper [16]. This formula was in fact identical to that of Black and Scholes, but depended on the expected rate of return of return of the underlying (that is μ in (3)) instead of the risk-free rate. In [5], Paul Samuelson derived a formula that also had unobservable parameters; the rate of expected return on the stock and the option, respectively. A few years later, formulas was also derived by Thorp and Kassouf in [19] in their book "Beat the Market". Thorp is an interesting case for several reasons. In [21], he claims that (comments in brackets)

[9]In the appendix, historic option pricing formulas has been included for the interested reader.

> ”I puzzled over the two parameters, M and d [*here M is the drift of the underlying price, μ in eq. (3) and d the discount factor in his formula*] and speculate that in a risk neutral world i can set them both equal to r [...] I didn't know how to proof the formula [*the BSM-formula*] but i decided to go ahead and use it to invest, because there was in 1967-68 an abundance of vastly overpriced over-the-counter options [*which he claimed based on his previous work*].”

This suggests, that he (and possibly others) knew the formula prior to Black-Scholes in 1973, but did not know how to proof it. What is also evident from this account is the fact, that he did not realize at the time, that pricing of derivatives did not require any assumptions on the preferences and risk-aversion of the agents.

3.4 Post-Black-Scholes: Risk-neutral Valuation

Cox, Ross 1976, [7], introduced the notion of risk-neutral valuation and preference independence formally. As the valuation is independent of the preferences assumed, one may as well assume that they are risk-neutral. In a risk-neutral world, all assets earns the risk-free rate in expectation which simplifies mathematics greatly, since (discounted) derivative payoffs, and the stock process, are martingales[10]. This implies, that one can price derivatives by calculating the risk-neutral, time-t conditional expectation of the payoff, so that the price of a call option is solved from:

$$C(S_t, t) = e^{-r(T-t)} E_t^Q(\max(S_T - K, 0)). \tag{8}$$

Here, Q is the risk-neutral measure[11]. Note the significance of this result: by operating under the risk-neutral measures, one can price derivatives as expected values of the payoff function. A theorem known as Girsanov's theorem allows one to switch between so-called equivalent martingale measures, see for instance [24] ch. 11.

The last piece of the puzzle was given by Harrison, Kreps and Pliska, in [14] and [15], of 1979 and 1981, respectively. Here, they effectively established the theory of complete and arbitrage-free markets using equivalent martingale measures and stochastic calculus. A huge amount of literature came to be in the subsequent years, but the foundation of mathematical pricing was established from hereon. It was now possible to specify the exact conditions under which a unique price for a derivative exists. A complete market is a market, in which every derivative can be replicated from trading in the underlying,

[10] A martingale is a stochastic process, X, which satisfies the property that $X_t = E^P[X_T|F_t]$, where P is the probability measure and F_t can be though of as the information up until, and including, time t.

[11] A measure can be thought of as a way of assigning probability to different states of the world.

also called an Arrow-Debreu market. If every derivative can be replicated, then every derivative has a price. Harrison, Kreps and Pliska established, what is now known as the 1st and 2nd fundamental theorem of asset pricing. These theorems state that a market is arbitrage if and only if an equivalent martingale measure exists and that an arbitrage-free market is complete if and only if the equivalent martingale measure is *unique*.

3.5 Nelson's "A B C of Options and Arbitrage"

We conclude this section with an account of a book on option pricing called "The A B C of Options and Arbitrage" written by a practitioner, Nelson, in 1904. While this chronologically belongs to the beginning of this section, we've included it in the end, as it seemed to be largely unknown in the formal literature on financial economics.

Nelsons account of how to price an option resembles the qualitative predictions of the BSM-formula to a certain degree. However, his account is heuristic, as no general theory or formulas are introduced in his work. Also, no reference was ever made to the work of Nelson in any of the major publications in financial economics, including Black, Scholes and Merton's papers. One must therefore logically conclude that either the book was considered unimportant in the scientific community or perhaps more plausible; it was simply unknown. The latter seems to be the most plausible explanation, as Haug claims to have re-discovered the book in the early 2000s, [27].

Nelsons book is interesting in relation to the question of how traders went about pricing options before 1973, but also to investigate the degree of understanding traders had of options at the time. On page 37 of [22], Nelson lists several properties that a successful option-trading business should take into account. For instance, he writes that it's important

> "to ascertain the past average fluctuation over a considerable period of time of the stock to be operated in",

which seems to be a statement on the importance of volatility, σ.

Another interesting point from Nelsons book is the following (in which he quotes Higgins from [23], written in 1902). Here, Nelson comments on the fact that markets tend to be more active in "calls" than in "puts", and writes that

> "[...] traders are more inclined to look at the bright side of things [...] This special inclination to buy 'calls' and to leave the 'puts' severely alone does not, however, tend to make 'calls' dear and 'puts' cheap."

This is clearly a statement on the put-call parity, formally credited to Stoll in [18] of 1969, some seventy years later. Since then, it has been discovered that other academics and practitioners knew of the parity long before the seventies. For instance, quite recently it was discovered that the Mathematics professor, Vinzens Bronzin, who himself did a lot of work on option pricing models as discussed in [25], derived the parity in 1908.

A central property of option trading is the knowledge of how much the option price moves in response to changes in the underlying. This is referred to as the delta. That is, the first-order sensitivity, or derivative in a mathematical sense, of the option price with respect to the underlying. In the BSM-formula in (2), the delta has a closed-form solution, and for at-the-money options (the strike, K, is equal to the value of the underlying, S_t) the delta is roughly 50%. This means, that if the underlying moves 1 unit of currency, the options value moves 0.5 units of currency. Apparently, this property was known by Nelson. On page 28, he writes, that

> "if they [*the sellers of options*] sell a call, [*they*] straightway buy half the stock against which the call is sold."

A seller of a call is exposed to a move in the underlying (i.e. he dislikes increases in the stock price). To protect himself, he simultaneously buy stocks for half the amount (a delta of 50%) to hedge his exposure to a stock price on the rise.

A last interesting point from his book, in relation to this paper, is from a section starting on page 20. Nelson gives an account of how one can hedge a "long position in a put option" dynamically, by subsequently buy more and more stocks to protect one against losses from increases in the stock price (if you own a put option, you want the stock to decrease in value). According to [27], this is effectively a statement on dynamic replication (the heart of Black Scholes derivation), and on page 21, Nelson argues, interestingly enough, that the approach works better in theory than in practice.

4 Discussion

We've now examined the history of option pricing, starting chronologically from Bachelier's thesis and Nelsons informal account on option trading. The latter served as evidence that practitioners, seventy years earlier than Black, Scholes and Merton's influential paper, had a good grasp on the fundamental properties of option pricing.

Bachelier, as we argued, were largely forgotten by the economics establishment until Samuelson rediscovered his thesis in the fifties. This does not mean, however, that he did not impact the course of the history of finance. For instance, Merton credits, and cites him in [3], for his contribution in the development of the mathematics of probability and stochastic calculus.

The major developments in asset pricing, such as the CAPM model, were similarly discussed, as was necessary to argue how BSMs approach was different in the sense that the preferences of individual investors did not matter in pricing derivatives. Then we moved on to discuss the progress in deriving an option pricing formula. Black, Scholes and Merton are perhaps most widely known for their formula, but as we discussed, the formula itself was already known. The difference was that Black and Scholes derived the formula without any dependence on unobservables, and at the same time they introduced the economic argument of arbitrage-free markets into option pricing. It's therefore quite clear, that the formula itself should not only be attributed to Black, Scholes and Merton.

Several researchers have historically worked on the option pricing problem starting with Bachelier and Bronzin. Later researchers, such as Samuelson, Boness, Thorp, Sprenkle, came very close in deriving the formula as it stands today. Also, the use of the geometric Brownian motion was an idea that Black and Scholes borrowed from elsewhere. On the other hand, it is also quite clear, that Black, Merton and Scholes jointly provided a satisfactory solution to the option pricing problem unlike any before them. The subsequent developments all revolved around Black, Scholes and Mertons formula, and equation, in an attempt to establish the conditions under which prices on derivatives could said to be unique and exist (and how to price them).

Another important aspect of Black, Scholes and Merton's contribution is the general framework on which their papers was based. Prior to them, research were occupied with pricing options in isolation. Their framework effectively lead to the realization, that one could price arbitrarily complicated payoff structures by the means of numerical methods, such as Monte Carlo or finite difference (to be developed after 1973), by changing the boundary condition in (4).

Lastly, one only has to read a few lines of the papers of Harrison, Kreps and Pliska ([14] and [15]) until Black and Scholes' paper is mentioned, but the real evidence of BSM's contribution comes from the fact, that these works formalizes the intuitive idea, implicit in Black-Scholes 1973 and explicit in Merton 1973, that one may calculate the prices on derivatives under the assumption of risk-neutrality. The final bricks of a thorough understanding in the field was laid with the discovery of the 1st and 2nd fundamental theorem of asset pricing, and the rest is history. With the final piece of the puzzle in place, mathematical finance was born as a separate research field and effectively separated from general equilibrium models and utility-maximization.

5 Conclusion

The focus of this paper has been on the Black, Scholes and Merton-model of 1973, but it's worth pointing out, that all three contributed widely to the field of finance. We concluded, that it was not the option price itself, but rather how it was derived, that was their main contribution, as it laid the foundation of a unified framework for pricing derivatives, independent of any assumptions of the preferences and risk-aversion of investors. Secondly, we've investigated the degree of understanding that both researchers and practitioners had of the central properties of option theory before 1973. Nelsons account suggests, that option traders had a fundamental understanding of how the price of an option is closely related to the movement in the underlying along with an understanding of advanced concepts such as dynamic hedging, the delta and the put-call parity. That being said, it's difficult to argue, based on Nelsons book, that Black, Scholes and Merton only formalized something, that was (somewhat) known.

6 Appendix - Option Pricing Formulas Through Time

In this section, we go through different attempts of deriving an call option pricing formula, mentioned in this paper, from Bachelier to Black-Scholes. Below, the BSM-formula has been written for convenience:

$$
\begin{aligned}
C &= N(d_1)S - Ke^{-rT}N(d_2) \\
d_1 &= \frac{\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}
$$

Here, S_t is the value of the underlying, T is time to maturity, K is the strike, r the "risk-free" interest rate, σ the volatility and $N(\cdot)$ the cumulative normal distribution. The notation has been modernized for easy comparison.

Bachelier, 1900, [4]:

$$
\begin{aligned}
C &= \sigma\sqrt{T}n(d) + (S_0 - K)N(d), \\
d &= \frac{S_0 - K}{\sigma\sqrt{T}}, \\
n(x) &= \frac{1}{\sqrt{2\sigma^2\pi}} \exp\left(- \frac{(x-u)^2}{2\sigma^2} \right).
\end{aligned}
$$

Here, $n(x)$ is the density of the normal distribution. Bachelier implicitly assumed that the interest rate was 0% in his thesis. This formula differs quite a lot from the other formulas. This is due to the assumption that the underlying follows a Brownian Motion - and not a geometric Brownian motion.

Sprenkle, 1961, [17]:

$$
\begin{aligned}
C &= Se^{\mu T}N(d_1) - (1-k)KN(d_2) \\
d_1 &= \frac{\ln(\frac{S}{K}) + (\mu + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}
$$

Here, k is an adjustment for the "degree of risk aversion". In a risk-neutral world, this parameter is 0.

James Boness (1964), [16]:

$$C = N(d_1)S - Ke^{-\mu T}N(d_2)$$
$$d_1 = \frac{\ln(\frac{S}{K}) + (\mu + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T},$$

Note, that this formula is identical to Black-Scholes, except that μ should equal r.

Paul Samuelson (1965), [5]:

$$C = Se^{\mu-\alpha}TN(d_1) - Ke^{-\alpha T}N(d_2)$$
$$d_1 = \frac{\ln(\frac{S}{K}) + (\mu + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T},$$

Here, α, is the average growth rate of the call option, as interpreted by Samuelson.

References

[1] The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel (1997). Nobelprize.org. Nobel Media AB 2014. Web. 6 Aug 2016. `http://www.nobelprize.org/nobel_prizes/economic-sciences/laureates/1997/`

[2] Black, Fischer and Scholes, Myron S, (1973). The Pricing of Options and Corporate Liabilities. Journal of Political Economy, University of Chicago Press, vol. 81(3), pages 637-54, May-June.

[3] Merton, Robert C., (1973). Theory of rational option pricing. Bell Journal of Economics and Management Science 4, 141-183.

[4] Bachelier, L., 1900. Theory of speculation. In: Cootner, P. (Ed.), (1964). The Random Character of Stock Market Prices. MIT Press, Cambridge.

[5] Samuelson, P., (1965). Rational theory of warrant pricing. Industrial Management Review 6, 13-31.

[6] Cox, John C., Ross, Stephen A. and Rubinstein, Mark, (1979). Option pricing: A simplified approach, Journal of Financial Economics, 7, issue 3, p. 229-263,

[7] Cox, John C. and Ross, Stephen A., (1976), The valuation of options for alternative stochastic processes, Journal of Financial Economics, 3, issue 1-2, p. 145-166,

[8] Duffie, Darrell, (1998), Black, Merton and Scholes - Their Central Contributions to Economics, Scandinavian Journal of Economics, 100, issue 2, p. 411-23.

[9] Markowitz, Harry, (1952), Portfolio Selection. Journal of Finance, 7, issue 1, p. 77-91.

[10] Sharpe, William, (1964), Capital asset prices: A theory of market equilibrium under conditions of risk. Journal of Finance, 19, issue 3, p. 425-442,

[11] Arrow, K. J., Debreu, G. (1954). Existence of an equilibrium for a competitive economy. Econometrica. 22 (3): 265-290.

[12] Modigliani, Franco, and Miller Merton H. "The Cost of Capital, Corporation Finance, and the Theory of Investment: Reply." The American Economic Review 49.4 (1959): 655-69.

[13] Milton Friedman, (1953). The Methodology of Positive Economics of his book "Essays in Positive Economics" of 1970, University of Chicago Press, p. 3-43.

[14] Harrison, J. Michael and Kreps, David, (1979), Martingales and arbitrage in multiperiod securities markets, Journal of Economic Theory, 20, issue 3, p. 381-408,

[15] Harrison, J. Michael and Pliska, Stanley R., (1981), Martingales and stochastic integrals in the theory of continuous trading, Stochastic Processes and their Applications, 11, issue 3, p. 215-260

[16] Boness, A., (1964). Elements of a theory of stock-option value. Journal of Political Economy 72, 163-175.

[17] Sprenkle, C., (1961). Warrant prices as indicators of expectations and preferences. Yale Economics Essays 1, 178-231.

[18] Stoll, Hans, (1969), The Relationship between Put and Call Option Prices, Journal of Finance, 24, issue 5, p. 801-24.

[19] Thorp, E.O., Kassouf, S.T., (1967). Beat the Market. Random House, New York.

[20] Thorp, E.O., (1969). Optimal gambling systems for favorable games. Review of the International Statistics Institute 37, 273-293.

[21] Thorp, E.O., (2002). What I knew and when I knew it - Part 1, Part 2, Part 3. Wilmott Magazine, Sep-02, Dec-02, Jan-03.
http://edwardothorp.com/id9.html

[22] Nelson, S.A., (1904). The A B C of Options and Arbitrage. The Wall Street Library, New York.

[23] Higgins, L.R., (1902). The Put-and-Call. E. Wilson, London.

[24] Bjork, Tomas, (2009), Arbitrage Theory in Continuous Time, 3 ed., Oxford University Press.

[25] Haug, E. G., (2009). The History of Option Pricing and Hedging. In: Hafner, Wolfgang. Zimmermann, Heinz (Ed.), 2009. Vinzenz Bronzin's Option Pricing Models: Exposition and Appraisal. Springer-Verlag Berlin Heidelberg

[26] Haug, E. G and Taleb, N N. Option Traders Use (very) Sophisticated Heuristics, Never the Black-Scholes-Merton Formula (February 26, 2009). Journal of Economic Behavior and Organization, Vol. 77, No. 2, 2011.

[27] Haug, E. G. Derivatives Models on Models, (2007). Wiley Finance.

[28] Davis, Mark, and Etheridge Alison. Louis Bachelier's Theory of Speculation: The Origins of Modern Finance, 183-89. Princeton University Press, (2006).